In presenting the dissertation as a partial fulfillment of the requirements for an advanced degree from the Georgia Institute of Technology, I agree that the Library of the Institute shall make it available for inspection and circulation in accordance with its regulations governing materials of this type. I agree that permission to copy from, or to publish from, this dissertation may be granted by the professor under whose direction it was written, or, in his absence, by the Dean of the Graduate Division when such copying or publication is solely for scholarly purposes and does not involve potential financial gain. It is understood that any copying from, or publication of, this dissertation which involves potential financial gain will not be allowed without written permission.

3/17/65
b

FATIGUE DAMAGE DUE TO WIDE-BAND RANDOM VIBRATION

A THESIS

Presented to

The Faculty of the Graduate Division

by

John Warner Shipley

In Partial Fulfillment

of the Requirements for the Degree

Master of Science in Engineering Mechanics

Georgia Institute of Technology

June, 1966

FATIGUE DAMAGE DUE TO WIDE-BAND

RANDOM VIBRATION

Approved:

Chairman

Date approved by Chairman: June 2, 1966

ACKNOWLEDGMENTS

The author wishes to thank Dr. Michael C. Bernard for his suggestion of the topic and his help throughout the preparation of this work. He also wishes to thank Dr. Eric R. Immel and Dr. David L. Finn for reading the manuscript.

This research was sponsored in part by the National Aeronautics and Space Administration, NASA Grant No. NsG-657.

TABLE OF CONTENTS

LIST OF ILLUSTRATIONS

SUMMARY

J. W. Miles' method of calculating fatigue damage for narrow-band random stresses is a standard method which is widely used in the aircraft industry. In the present research the general problem of fatigue damage done by random stresses is considered, and an approach is used which does not depend on the narrow-band assumption that Miles employed.

The classical concept of cumulative damage originally postulated by M. A. Miner is reviewed. Then using this concept and the half-wave design method proposed by W. L. Starky and S. M. Marco for evaluating fatigue damage due to complex deterministic stress records, a damage function is developed from which an estimate (lower bound) of fatigue life can be made. From this damage function, a method of calculating fatigue damage for a random process is derived.

It is shown that for the case of a narrow-band, stationary, normal stress process the expression derived here agrees with the expression derived by J. W. Miles based on the narrow-band assumption. Also, it is noted, this expression compares favorably with the peak stress criterion proposed by A. K. Head and F. H. Hooke for calculating damage for a random stress process.

CHAPTER I

INTRODUCTION

The study of the properites of materials and the geometry of structures durino the past few centuries has led to design criteria for predicting failure in structures due to static and dynamic loading. These criteria are based on such material properties as yield strength and ultimate strength, i.e., they indicate that a structure will fail when the ultimate strength of any member is exceeded. In the last sixty years the development of rotating machinery such as high-speed turbines has shown that design criteria such as these are, in many situations, no longer adequate for the determination of allowable loads, since in such applications members which are never stressed beyond their elastic limits will nevertheless fail when a sequence of loadings is applied over a long time. This type of failure is called "fatigue failure."

Much recent research, both theoretical and experimental, has been devoted to the development of criteria for fatigue failure. Nevertheless, there is at present no general formulation, based substantially on material characteristics, which accurately predicts fatigue failure. A reasonable good qualitative explanation of the micro-mechanism of fatigue failure does exist. Researchers as early as Grough in 1926 [1]* have discussed the effect of the crystalline nature of a material on its fatigue life. Subsequently many others, such as Dolan, Lazan, and Horger [2] in their book of fatigue in 1953, have considered fatigue life from this

*Numbers in brackets refer to items in the Literature Cited section.

viewpoint. They explain the occurrence of fatigue in terms of such properties as slip reversals, molecular dislocations, and material imperfections. However, the theories developed along these lines do little more than show why damage accumulates in structural materials. Other investigators such as Langer in 1937 [3] and Miner in 1945 [4] have considered the problem from a different viewpoint. They have proposed empirical relations for the accumulation of damage which have been, in the absence of anything better, quite useful in predicting the fatigue life of design parts subject to cyclic loading. Much of this work is based on a concept of damage accumulation which postulates that the damage done in each loading cycle is a function of the stress amplitude at some point in the structure and the damages from individual cycles add in a linear manner. When this sum equals unity, failure will occur. By plotting the number of loading-cycles-to-failure for various stress amplitudes, a curve can be obtained from which the amount of damage done in one loading cycle at a given amplitude can be determined. Then the fatigue life of a structure which encounters a sequence of cycles of various amplitudes can be predicted by equating the sum of the damages done in individual cycles to unity.*

The determination of the damage incurred by a structure which is placed in a random environment, i.e., a turbine blade in a jet engine, is in general quite a different matter. The force applied to a structure in this case does not consist of cycles of constant amplitude but is characterized by a continuous frequency spectrum and a probabilistic distribution of amplitudes. A typical frequency spectrum might have its energy

*This is a brief summary of Miner's criterion; a more detailed explanation is given in Chapter II.

distributed in a frequency band as wide as 2000 cps [5]. Hence a structure with more than one mode of vibtation will be excited into a motion which is not simple harmonic [6]. In this event a cyclic loading fatigue criterion cannot be used to calculate the fatigue life of a structure.

To add to the complexity of this problem, due to the natureof most random phenomena, time histories, recorded under essentially uniform conditions, will appear to be quite different functions. However their statistical parameters may well be nearly the same. Thus a particular record of a random phenomenon, i.e., a record of stress in a wing spar during a flight from Atlanta to Chicago, may be considered as a sample function of a wide class of functions, namely a stochastic process. Therefore, the response of a structure must be discussed in terms of the statistical properties of the forcing function process rather than in terms of a particular record of the forcing function.

The problem that will be discussed here involves the formulation of a general approach which can be employed to determine the fatigue life of a structure excited by a sample function of a random process, such as a jet noise record. This is clearly a question of great importance to the aircraft industry. Design criteria for structural parts cannot be determined with great confidence without some knowledge of fatigue conditions in a jet noise environment or fatigue damage resulting from gust loads.

The calculation of the fatigue damage of a structure which has a narrow-band* response to noise loading is a special case of this problem

---

* The meaning of "narrow-band" will be described in Appendix I; for this part of the discussion it is sufficient to say that a random process is narrow-band if its sample functions appear as sinusoids with slowly varying amplitude and phase.

which has been discussed by J. W. Miles [7], W. D. Mark [8], and others. Compared with the general problem its analysis is greatly simplified by the fact that a displacement record and hence a stress record will appear sinusoidal except for a varying phase and amplitude. A cyclic loading criterion can be applied here by taking into account the distribution of stress amplitudes. The analysis of this special case is somewhat limited in application however, since a narrow-band response only occurs when one of the following conditions is satisfied:

1. The structure is lightly damped and has only one degree of freedom.
2. The structure is lightly damped and the power of the forcing function is concentrated mostly about the frequency of the first mode of the structure.

The general result which is derived in Chapter III does not require this sinusoidal character of the structural response. Therefore, it can be applied to a more general class of problems. This method by no means completely resolves the problem in that it does not give an exact value for the expected fatigue life of a structure. However, it does give an estimate which in some cases has been shown to be quite reasonable.

CHAPTER II

THE CONCEPT OF CUMULATIVE DAMAGE

In Chapter III Miner's criterion for the accumulation of fatigue damage is extended in order to discuss fatigue damage for wide-band random stresses. The purpose of this chapter is to introduce Miner's criterion and to review some common extensions of it which are now used, since this is the basis of the work in Chapter III.

Miner's Criterion

Miner's hypothesis asserts that if a structure is loaded in a sinusoidal manner*, failure will occur at that point in the structure where the stress level is a maximum and when the summation of the increments of damage** equals unity.

From experiments run on simple structures, Fig. 1, under constant stress amplitude, for a particular amplitude a it is found that a failure will occur after a certain number of cycles, designated N(a), the number-of-cycles-to-failure for stress amplitude a. By testing at different amplitudes a curve relating number-of-cycles-to-failure to stress

* In his original work Miner postulated only strictly sinusoidal loading. One might postulate just cyclic loading here, but the assumption that exactly the same damage would be obtained from a square wave as from a triangular wave is one which would require experimental verification.

**An increment of damage is the damage which is associated with one cycle and is a function of the amplitude of the cycle. Since N(a) is defined in the next paragraph as the number of cycles to failure at amplitude a, the increment of damage associated with this stress level is 1/N(a).

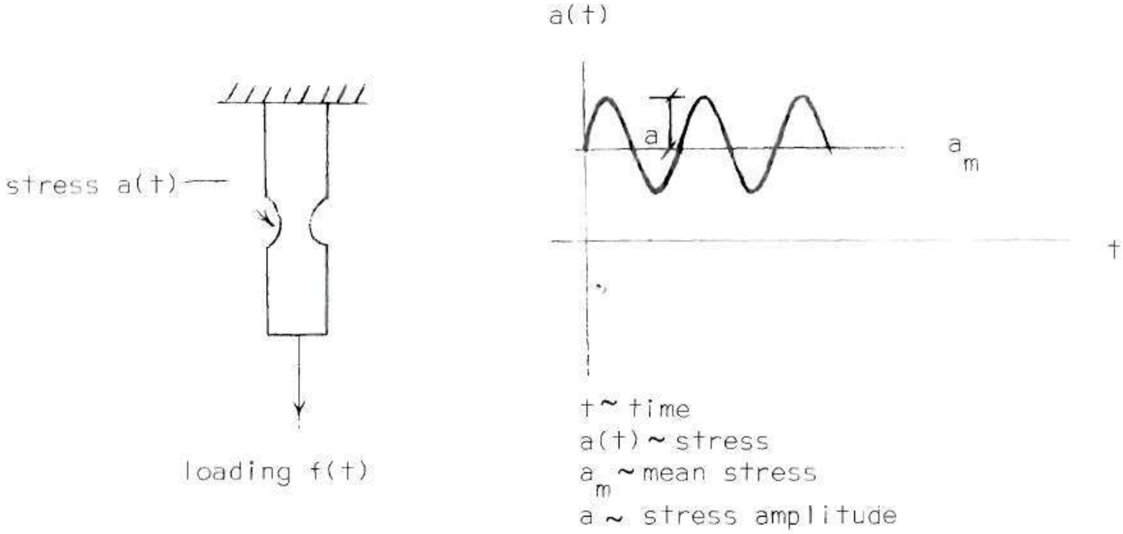


Figure 1. Typical Structure and Its Stress Record.

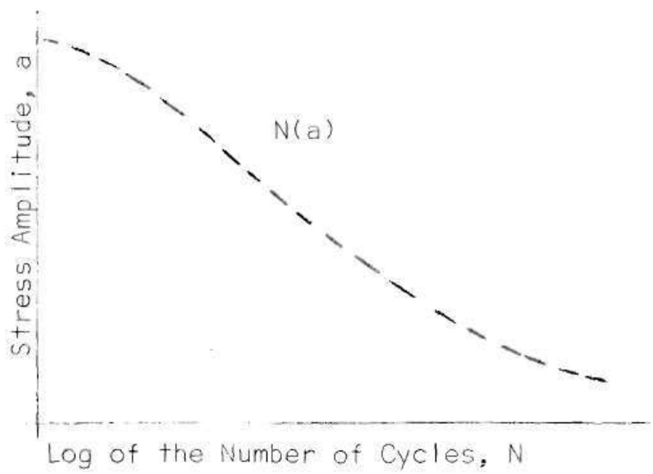


Figure 2. Stress Versus Number-of-Cycles-to-Failure Curve.

amplitude can be obtained , Fig. 2.

By using the curve in Fig. 2, the fatigue damage due to a sequence of sinusoidal loadings of different amplitudes can be calculated [4]. For example, if a stress record consists of h_1 cycles of amplitude a_1 and h_2 cycles at amplitude a_2 then the fraction of total damage done by this record is $h_1/N(a_1) + h_2/N(a_2)$. A failure will occur for a sequence of such loadings when the sum of the corresponding fractions equals unity.

Damage for Randomly Modulated Sinusoids

If a stress record is a member function of a random process, it is possible to calculate the expected fatigue life if the following conditions are satisfied**:

1. The stress process $\{A(t)\}$ can be represented by $\{A^*(t)\sin w_o t\}$ where w_o is a fixed frequency or a random process with small variance, and $\{A^*(t)\}$ is a random process which is characterized by member functions which vary slowly in comparison with $\sin w_o t$.***
2. $\{A^*(t)\}$ is stationary and ergodic so that an ensemble average for the process will be equal to a time average calculated from a sample function.

---

* Engineers in the latter part of the nineteenth century used such curves to predict constant amplitude fatigue life; see the introduction to Ref. 1. Not until Miner's work in 1945 was any concept of accumulation of damage widely accepted for varying amplitude sinusoidal loads.

** The analysis here is essentially the same as that presented by F. A. McClintock [9].

*** $\{A^*(t)\}$ is a random process which modulates the sinusoid.

If the probability density function for the random variable $A^*(t)$ is $f_A^*(a;t)$ and this is constant with time, then it can be written as $f_A^*(a)$, and the probability that the random variable $A^*(t)$ is between a and a + da is $f_A^*(a)$ da. The number of cycles per unit time of the sinusoid is $w_o/2\pi$. Since $A(t)$ takes on the value $A^*(t)$ when $\sin w_o T$ is + 1, once each cycle, the expected number of cycles per unit time with amplitudes between a and a + da is $w_o/2\pi f_{A^*}(a)$ da. The fraction of total damage done by $w_o/2\pi f_{A^*}(a)$ da cycles is $w_o/2\pi f_{A^*}(a)\ 1/N(a)$ da.* To obtain the expected fraction of total damage per unit time, this expression must be integrated over all possible values of $A^*(t)$ to obtain

$$E[D(t)]^{**} = \frac{w_o}{2\pi} \int_{a_{min.}}^{a_{max.}} \frac{f_{A^*}(a)}{N(a)}\, da \qquad (1)$$

The time, T, found by solving the equation,

$$E[D(t)] \cdot T = 1 \qquad (2)$$

is the expected time to failure.

For some materials the stress versus loading-to-cylces-to-failure curve, Fig. 2, can be approximated by a straight line on a log-log plot, Fig. 3. This curve can then be represented by the equation,

$$\log a = \log a_1 - \frac{1}{b} \log N(a) \qquad (3a)$$

---

*See footnote** on page 6.

**$E[D(t)]$ will be defined in chapter III as the expected rate of accumulation of damage. Thy symbol is used here for convenience only.

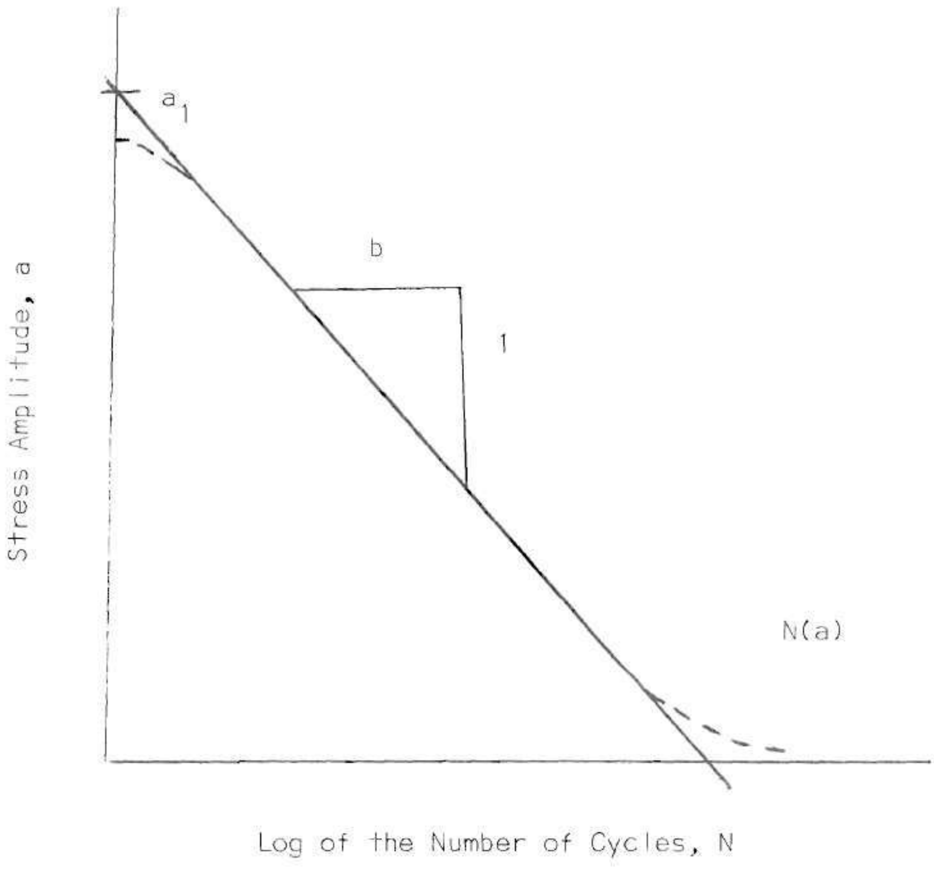


Figure 3. Log - Log Approximation of Fatigue Curve

which can be rewritten as

$$N(a)\ a^b = a_1^b \tag{3b}$$

where b and a_1^b are constants which depend on the material. For convenience let a_1^b equal C. The damage for one cycle, $1/N(a)$, can be written as a^b/C, and equation (1) becomes

$$E\ D(t) = \frac{w_o}{2\pi} \int_{a_{min.}}^{a_{max.}} \frac{a^b}{C} f_{A*}(a) \cdot da \tag{4}$$

This formula can be used to calculate the expected damage for a narrow-band stress process. See Appendix I.

The Applicability of Miner's Criterion

While in the absence of anything better Miner's criterion is widely accepted and widely used, it is somewhat artificial. It does not include all the variables which influence fatigue life, and it does not consider a very general type of problem. Firstly, its basic hypothesis is that a stress record is sinusoidal. Certainly, this limits its application. Specifically, the problem considered in this thesis is outside the range of a direct application of Miner's criterion. Secondly, it is completely independent of frequency. If two identical structures are excited by different frequency loadings at the same amplitude, the predicted accumulation of damage for both will be the same for the same number of loading cycles. The fact that loading frequency has at least a slight influence on fatigue life is well known and is discussed by Wade and Groutenhuis [10] and Lomas, Ward, Rait, and Colbeck [11]. Thirdly, it does not allow for changes in the order of loading. For some metals, if a sinusoidal stress below the limit of endurance* is applied for a period of time, the fatigue life of the metal is increased; however, if a high level loading were applied first, even loading below the limit of endurance would cause damage to accumulate. Also, Miner's criterion does not take into account other factors which could influence fatigue damage such as residual stresses. Therefore, it should not be surprising that fatigue data often possesses wide statistical scatter.

In spite of many faults Miner's criterion is very useful in an engineering sense because of its simplicity and the fact that so many metals conform to predictions based on it in at least a gross sense. It is not the purpose of this work to develop a new fatigue criterion. The object is instead to extend Miner's criterion along the lines of the work of Starky and Marco [12] for complex stresses in order to discuss fatigue for a general wide-band random process.

Damage Due to Complex Deterministic Stress Records

It was pointed out in the last section that the accumulation of damage due to stress records which are not sinusoidal cannot be calculated by directly applying Miner's criterion. If such a criterion is to be used at all, a plausibility argument must be employed to convert the complex stress record to a sequence of sinusoids which would produce an equivalent amount of damage. There are many possible approaches which could be used. For example, records could be decomposed into a Fourier series, if periodic, and it could be postulated that the damage incurred is equivalent to the sum of the damages that would be incurred by each component of the series. Alternatively, the damage could be postulated

*For some materials there exists a particular value of stress, such that, if this value is never exceeded, no fatigue damage will accumulate.

to be equivalent to the damage that would be accumulated by half sine waves with double amplitudes equal in value to the absolute differences between pairs of successive extreme and mean values equal to the mean values of the paris of successive extrema.

Any such approach would have to be interpreted as an intuitive extension of a sinusoidal loading criterion and would have to be verified experimentally before much confidence could be placed in it. The last approach mentioned is the half-wave design method developed by Starky and Marco. Since this method describes more accurately than other methods the accumulation of fatigue damage and since it has at least been partially verified by experiment [12]*, it will provide the basis for the work done here even though it relies quite heavily on empirical formulae.

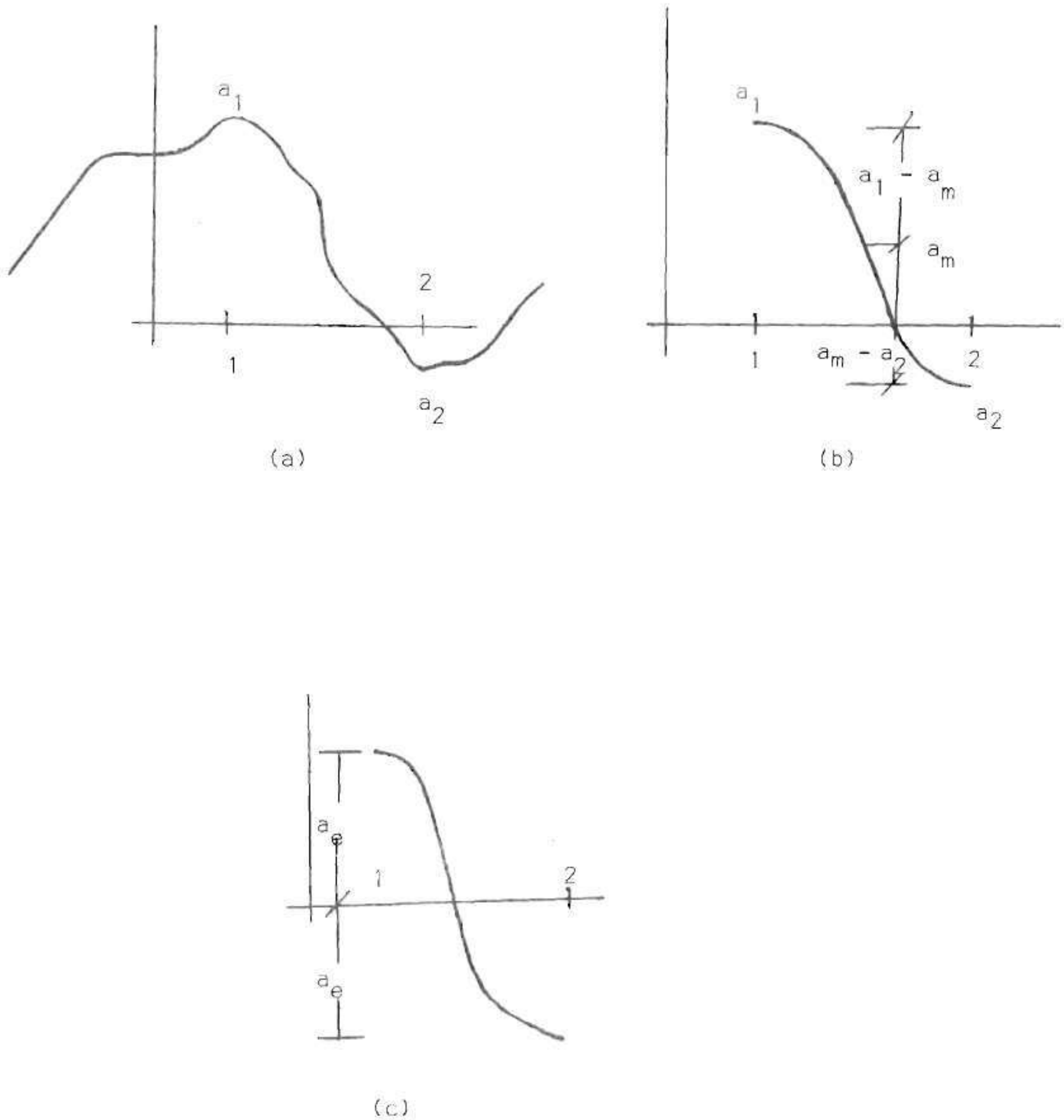
The Half-Wave Design Method

The principle of the half-wave design method has already been described. To predict the damage for an arbitrary stress record, consider the record in Fig. 4a. The damage done between peaks 1 and 2 is, according to this method, assumed to be equal to the damage done by the sinusoid in Fig. 4b. The damage done by this sinusoid is postulated to equal the damage done by the sinusoid in Fig. 4c with zero mean and a new amplitude, called an equivalent amplitude, a_e, determined by one of the following relationships:

1. Modified Goodman law (tension mean)

$$a_e = \frac{a_1 - a_m}{1 - \frac{a_m}{a_u}}, \quad a_m > 0,\ a_1 < a_{y.p.} \tag{5a}$$

*See Appendix III



a_1 ~ larger extrema

a_2 ~ smaller extrema

a_e ~ equivalent zero mean stress

Figure 4. Equivalent Fatigue Conversion

2. Gerber law (tension mean)

$$a_e = \frac{a_1 - a_m}{1 - \left[\frac{a_m}{a_u}\right]} \quad , \quad a_m > 0, \ a_1 < a_{y.p.} \tag{5b}$$

3. Compression mean*

$$a_e = a_m - a_2 \ , \qquad a_m < 0, \ a_2 > a_{y.p.} \tag{5c}$$

Here a_u is the ultimate stress, and $a_{y.p.}$ is the yield point stress for a given material. The selection of which tension mean relationship to use for an actual fatigue calculation depends on the material considered. The equivalent amplitude given by the modified Goodman law will assess more damage. Since the object of this work is to obtain a conservative estimate for fatigue life and since for most common materials the Gerber law appears to assess too little damage while the Goodman law appears to assess excessive damage, the modified Goodman law will be used for the remainder of the discussion.

The damage for a complex stress wave can be obtained by summing the damages done by the equivalent half-sine-cycles. Since the damage for one cycle of a sinusoid of amplitude a is $\frac{a^b}{C}$, assuming the relationship shown in Fig. 3, this sum expressed in terms of the equivalent amplitude is

$$\text{damage} = \frac{1}{2C}(a_{e_1}^b + a_{e_1}^b + \ldots + a_{e_n}^b) \tag{6}$$

*These relationships are used to account for the fact that the mean stress has an effect on the fatigue life of a material. If a specimen is in tension, more damage is done by one cycle of a given amplitude than would be done if the mean stress were zero. If the mean stress is compression, the same damage is done as would be done if the mean stress were zero. They were originally obtained by curve fitting on constant sine fatigue test data. See O'Connor, Morrison, and Mech [13].

The subscripts on the a_e's represent the successive extrema to which the equivalent amplitudes correspond.

CHAPTER III

CALCULATION OF FATIGUE DAMAGE FOR WIDE-BAND RANDOM STRESSES

The object of this chapter is to set up an expression for the damage accumulated when the stress record is a member function of a stochastic process [14]. The method used is based upon the half-wave design method which is first modified slightly for the convenience of later statistical considerations.

A Modification of the Half-Wave Design Method

First, the stress versus loading-cycles-to-failure-curve in Fig. 3 will be postulated. Then for a record $a^i(t)$ of a random stress process, Fig. 5, the damage incurred in the time interval (T_1, T_2) can be calculated by using equation (6) in chapter II to obtain

$$\text{damage} = \frac{a_{e_1}^b}{2c} + \frac{a_{e_2}^b}{2c} + \ . \ . \ . + \frac{a_{e_7}^b}{2c} \tag{1}$$

Note that there is one equivalent amplitude for each pair of successive extrema. The equivalent amplitude a_{e_1} which corresponds to the half-cycle associated with the extrema $a^i(t_1)$ and $a^i(t_2)$ at times t_1 and t_2 is, according to equation (5) of chapter II,

$$a_{e_1} = \frac{a^i(t_1) - a_{m_1}}{1 - \frac{a_{m_1}}{a_u}} \tag{2}$$

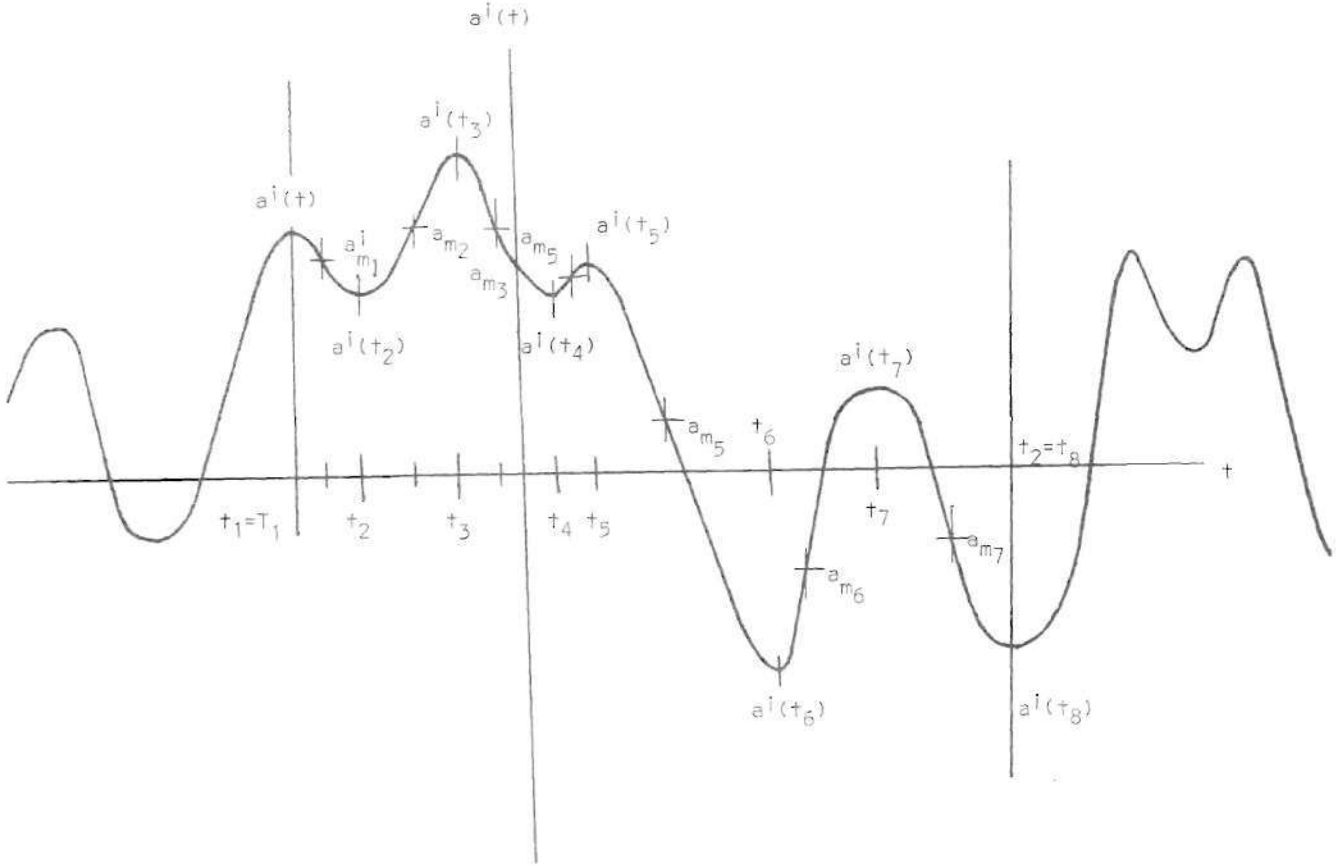


Figure 5. An Arbitrary Record of a Random Stress Process.

(The a_{m_n}'s are the means between sucessive extrema,

and the $a^i(t_n)$'s are the extrema.)

Similarly, the other equivalent amplitudes are

$$\left.\begin{aligned}
a_{e_2} &= \frac{a^i(t_3)-a_{m_2}}{1-\dfrac{a_{m_2}}{a_u}} \\[2ex]
a_{e_3} &= \frac{a^i(t_3)-a_{m_3}}{1-\dfrac{a_{m_3}}{a_u}} \\[2ex]
a_{e_4} &= \frac{a^i(t_5)-a_{m_4}}{1-\dfrac{a_{m_4}}{a_u}} \\[2ex]
a_{e_5} &= \frac{a^i(t_5)-a_{m_5}}{1-\dfrac{a_{m_5}}{a_u}} \\[2ex]
a_{e_6} &= a_{m_6} - a^i(t_6) \\[2ex]
a_{e_7} &= a_{m_7} - a^i(t_8)
\end{aligned}\right\} \qquad (3)$$

Since a_{m_1} is the mean value of $a^i(t_1)$ and $a^i(t_2)$, $a^i(t_1) - a_{m_1}$ is equal to $a_{m_1} - a^i(t_2)$; and equation (2) can also be written as

$$a_{e_1} = \frac{a_{m_1} - a^i(t_2)}{1-\dfrac{a_{m_1}}{a_u}} \qquad (4)$$

Similar expressions can be obtained for all seven equivalent amplitudes. The equivalent amplitudes listed as equations (2) and (3) are now substituted into equation (1), which after a slight rearrangement of terms convenient for the subsequent work, gives the following expression for the damage:

$$\text{damage} = \frac{1}{2C}\left\{\begin{aligned}&\left|\frac{a^i(t_1)-a_{m_1}}{1-\frac{a_{m_1}}{a_u}}\right|^b + \left|\frac{a_{m_2}-a^1(t_3)}{1-\frac{a_{m_2}}{a_4}}\right|^b + \left|\frac{a^i(t_3)-a_{m_3}}{1-\frac{a_{m_3}}{a_u}}\right|^b \\ &+\left|\frac{a_{m_4}-a^i(t_5)}{1-\frac{a_{m_4}}{a_u}}\right|^b + \left|\frac{a^i(t_5)-a_{m_5}}{1-\frac{a_{m_5}}{a_u}}\right|^b + \left|a^i(t_6)-a_{m_6}\right|^b + \left|a_{m_7}-a^i(t_8)\right|^b\end{aligned}\right\} \qquad (5)$$

Using equivalent amplitudes of the form in equation (4), a similar expression can be obtained. It is

$$\text{damage} = \frac{1}{2C}\left\{\begin{aligned}&\left|\frac{a_{m_1}-a^i(t_2)}{1-\frac{a_{m_1}}{a_u}}\right|^b + \left|\frac{a^i(t_2)-a_{m_2}}{1-\frac{a_{m_2}}{a_u}}\right|^b + \left|\frac{a_{m_3}-a^i(t_4)}{1-\frac{a_{m_3}}{a_u}}\right|^b \\ &\left|\frac{a^i(t_4)-a_{m_4}}{1-\frac{a_{m_4}}{a_u}}\right|^b + \left|\frac{a_{m_5}-a^i(t_6)}{1-\frac{a_{m_5}}{a_u}}\right|^b + \left|a_{m_6}-a^i(t_7)\right|^b + \left|a^i(t_7)-a_{m_7}\right|^b\end{aligned}\right\} \qquad (6)$$

Summing equations (5) and (6) and dividing by two, the following expression for the damage is obtained:

$$\text{damage} = \frac{1}{4C}\left\{\begin{aligned}&\left|\frac{a^i(t_1)-a_{m_1}}{1-\frac{a_{m_1}}{a_u}}\right|^b + \left|\frac{a_{m_1}-a^i(t_2)}{1-\frac{a_{m_1}}{a_u}}\right|^b + \left|\frac{a^i(t_2)-a_{m_2}}{1-\frac{a_{m_2}}{a_u}}\right|^b + \left|\frac{a_{m_2}-a^i(t_3)}{1-\frac{a_{m_2}}{a_u}}\right|^b \\ &+\left|\frac{a_{m_3}-a^i(t_3)}{1-\frac{a_{m_3}}{a_u}}\right|^b + \left|\frac{a_{m_3}-a^i(t_4)}{1-\frac{a_{m_3}}{a_u}}\right|^b + \left|\frac{a^i(t_4)-a_{m_4}}{1-\frac{a_{m_4}}{a_u}}\right|^b + \left|\frac{a_{m_4}-a^i(t_5)}{1-\frac{a_{m_4}}{a_u}}\right|^b \\ &+\left|\frac{a^i(t_5)-a_{m_5}}{1-\frac{a_{m_5}}{a_u}}\right|^b + \left|\frac{a_{m_5}-a^i(t_6)}{1-\frac{a_{m_5}}{a_u}}\right|^b + \left|a^i(t_6)-a_{m_6}\right|^b + \left|a_{m_6}-a^i(t_7)\right|^b \\ &+\left|a^i(t_7)-a_{m_7}\right|^b + \left|a_{m_7}-a^i(t_8)\right|^b\end{aligned}\right\} \qquad (7)$$

The expected damage in the interval (T_1, T_2) could be found by calculating a value for equation (7) for each member stress function and performing an ensemble average. An analytical procedure would, however, rely quite heavily on the distributions of extrema; and such problems as yet remain largely unsolved [15].

A useful estimate of the damage given by equation (7) can be calculated. For this purpose it is convenient to employ a functional $f(a^i(t))$ of the stress process defined by

$$f(a^i(t)) = \begin{cases} (a^i(t))^b , & a^i(t) \geq 0 \\ -|a^i(t)|^b , & a^i(t) < 0 \end{cases} \tag{8}$$

The choice of this particular functional is logical as will be shown in the subsequent work.

For the present note that

1. if the original record $a^i(t)$ is differentiable, e.g., boundedly differentiable then so is the function given in expression (8), and its derivative is

$$\frac{d\, f(a^i(t))}{dt} = b \left| a^i(t) \right|^{b-1} \frac{d\, a^i(t)}{dt} \tag{9}$$

2. $f(a^i(t))$ will have maxima and minima for the same values of t as $a^i(t)$.*

Further, it is shown in appendix II that the inequality

$$\left| f(a^i(t_n)) - f\,(a^i(t_{n+1})) \right| \geq \left| a^i(t_n) - a_{m_n} \right|^b + \left| a_{m_n} - a^i(t_{n+1}) \right|^b \tag{10}$$

---

* These observations are discussed in appendix II.

always holds; and that the inequality

$$|f(a^i(t_n)) - f(a^i(t_{n+1}))| \geq \left|\frac{a^i(t_n)-a_{m_n}}{1-\frac{a_{m_n}}{a_u}}\right|^b + \left|\frac{a_{m_n}-a^i(t_{n+1})}{1-\frac{a_{m_n}}{a_u}}\right|^b \qquad (11)$$

will hold at least approximately under restrictions imposed by reasonable design methods.

Using these observations, a relation between the damage calculated by the half-wave design method and the functional $f(a^i(t))$ can be formed. For example, for the record in Fig. 5, if each term in equation (7) is replaced by the dominating term defined by inequality (10) or (11), the relation becomes

$$\text{damage} \geq \frac{1}{4C} \sum_{n=1}^{7} |f(a^i(t_n)) - f(a^i(t_{n+1}))| \qquad (12)$$

Since for an arbitrary record the damage as calculated by the half-wave design method for any time interval (T_1, T_2) with extrema, $a^i(t_1)$, $\ldots a^i(t_k)$, will be a sum of the form of equation (7), a general expression corresponding to inequality (12) is

$$\text{damage} \geq \frac{1}{4C} \sum_{n+1}^{k-1} |f(a^i(t_i)) - fa^i(t_{n+1})| \qquad (13)$$

$f(a^i(t))$ will henceforth be called the damage function.

As just mentioned, since the $a^i(t_n)$'s are the extreme values of the function $a^i(t)$, the $f((a^i(t_n))$'s are the extreme values for the function $f(a^i(t))$. Therefore, the sum in inequality (13) corresponds to the total variation* of the function $f(a^i(t))$ in the interval whose end points are

* See Apostol [16] Ch. 8.

t_1 and t_k, the times corresponding to the first extremum and the last extremum. Because the $f(a^i(t))$'s under consideration have bounded first derivatives, the total variation of f on a^i in the interval (t_1, t_k), $V_{foa^i}(t_1, t_k)$, can be written as

$$V_{foa^i}(t_1, t_k) = \int_{t_1}^{t_k} \left| \frac{df\ a^i(t)}{dt} \right| dt \tag{14}$$

Integrating this function over the interval (T_1, T_2) would make only a negligible change in the value of the integral when considering time intervals as large as the fatigue life of a metal because for such large intervals t_1 and t_k, the times associated with the first and last extrema, will be very close to T_1 and T_2. Therefore, the damage incurred in a time interval (T_1, T_2) is

$$\text{damage} \geq \frac{1}{4C} \int_{T_2}^{T_1} \left| \frac{df\ a^i(t)}{dt} \right| dt \tag{15}$$

The above inequality suggests that $\frac{1}{4C} \left| \frac{df(a^i(t))}{dt} \right|$ is the rate of accumulation of damage for the process, at least in the sense that the time integral of this expression gives a bound for the damage in a given time. This expression will be called $D^i(t)$, the rate of accumulation of damage on the i'th record.

The Expected Damage for a Time Interval

For a random stress process $\{A(t)\}$ which is differentiable in the mean-square sense a derivative process $\{\frac{d(A(t)}{dt}\}$ exists in the mean-square sense [14]. Then, for a fixed time t, if the random variables $A(t)$ and $\dot{A}(t)$ associated with the stress process and the derivative of the stress process, respectively, possess a joint probability density function $f_{A\dot{A}}(a, \dot{a};t)$, it follows that $E[D(t)]$ the ensemble average of the rate of accumulation of damage $D^i(t)$ is given by

$$E[D(t)] = \frac{1}{4C}\iint_{-\infty}^{\infty} b|a|^{b-1}|\dot{a}|\ f_{A\dot{A}}(a, \dot{a};t)\ da\ d\dot{a} \qquad (16)$$

Note that from page

$$D^i(t) = \frac{1}{4C}\left|\frac{dfa^i(t)}{dt}\right| = \frac{1}{4C}b\ a|(t)|^{b-1}\left|\frac{da^i(t)}{dt}\right| \qquad (17)$$

The expected damage done in a time interval (T_1, T_2) is by equation (15)

$$E\left[\text{damage } (T_1, T_2)\right] \geq \int_{T_1}^{T_2} E[D(t)]\ dt \qquad (18)$$

To give inequality (18) a useful interpretation in calculating the probable life of an individual structural part, an additional assumption must be made. This is that the process is stationary and ergodic. Then since the operator $\frac{1}{4C}\frac{df(x)}{dt}$ is a time invariant filter, the output process, whose sample functions are the $D^i(t)$'s, is a stationary ergodic random process [17]. Thus, for reasonably long time intervals such as would be encountered in fatigue life predictions, the bounds given by inequalities (15) and (18) will be equal.

It should be noted that no assumption has been made regarding the band width of the stress process in deriving inequality (18); thus the relation will apply to any stress process, subject to suitable differentiability conditions.

Expected Rate of Accumulation of Damage for a Normal Stress Process

Noise random forcing functions, such as acoustical noise and jet noise, are approximately normal processes [18], and in practical applications a great many structures are linear or approximately linear. When the input to a linear system, in this case a force, is a normal process, the output will also be normal [19], in this case the stress. Therefore, the consideration of fatigue damage for a normal stress process is of great importance.

For a normal stationary stress process $A(t)$, which has zero mean, variance σ_A^2, and a covariance $\Gamma_{AA}(\tau)$ which has a second derivative, the joint density function for the random variable $A(t)$ associated with the stress process at time t and the random variable $\dot{A}(t+\tau)$ associated with the rate of change of stress process at time $t + \tau$ is a normal joint density function [15] given by

$$f_{A\dot{A}}(a,\ t;\ \dot{a},\ t+\tau) = \frac{1}{2\pi\sqrt{\sigma_A^2\ \sigma_{\dot{A}}^2 - \Gamma_{A\dot{A}}(\tau)}}\, e^{-\frac{1}{2}\left[\frac{a^2\sigma_{\dot{A}}^2 + 2\dot{a}a\Gamma_{A\dot{A}}(\tau) + \dot{a}^2\sigma_A^2}{\sigma_A{}^2\ \sigma_{\dot{A}}{}^2\ \ \Gamma_{A\dot{A}}(\tau)}\right]} \tag{19}$$

$\Gamma_{A\dot{A}}(\tau)$ is the cross-covariance between $A(t)$ and $\dot{A}(t+\tau)$, and it equals $\frac{d}{dt}\Gamma_{AA}(\tau)$. $\sigma_{\dot{A}}^2$ is the variance of $\dot{A}(t)$, and it equals $-\frac{d}{dt^2}\Gamma_{AA}(\tau)\Big|_{\tau=0}$. The joint density function needed to evaluate $E[D(t)]$ can be obtained from equation (19) by letting $\tau=0$. Then, since $\Gamma_{A\dot{A}}(0) = 0$ [15], this density function is

$$f_{A\dot{A}}(a,\dot{a};\ t) = \frac{1}{2\pi\sigma_A\sigma_{\dot{A}}}\, e^{-\frac{1}{2}\left[\frac{a}{\sigma_A}^2 + \frac{\dot{a}}{\sigma_{\dot{A}}}^2\right]} \tag{20}$$

Once this density function is known the expected rate of accumulation of damage can be calculated from

$$E[D(t)] = \frac{1}{4C} \int\!\!\int_{-\infty}^{\infty} b|a|^{b-1} |\dot{a}| \frac{1}{2\pi\sigma_A\sigma_{\dot{A}}} e^{-\frac{1}{2}\left[\frac{a^2}{\sigma_A{}^2} + \frac{\dot{a}^2}{\sigma_{\dot{A}}{}^2}\right]} da\,d\dot{a} \tag{21}$$

Upon simplifying, equation (21) becomes

$$E[D(t)] = \frac{\sigma_{\dot{A}}{}^2}{2\pi\sigma_A\sigma_{\dot{A}}} \int_0^{\infty} b\ a^{b-1}\ e^{-\frac{1}{2}\frac{a^2}{\sigma_A{}^2}}\ da \tag{22}$$

This expression can be evaluated by the use of the gamma function. The expected damage that would occur in a time interval (T_1, T_2) will be less than or equal to $E[D(t)]\ (T_2 - T_1)$.

CHAPTER IV

DISCUSSION OF THE RESULT

Comparison of the Fatigue Damage Calculation in Chapter III With the Narrow-Band Fatigue Calculation

Equation (9) of Appendix I is the expected rate of accumulation of damage, based on the method originally used by Miles [7], for a narrow-band normal stress process.* To compare the expected rate of accumulation of damage derived in Chapter III for a normal stress process, equation (22) with equation (9) of Appendix I observe that by integrating equation (22) by parts twice $E[D(t)]$ becomes

$$E[D(t)] = \frac{\sigma_{\dot{A}}}{2C\pi\sigma_A \sigma_A{}^2} \int_0^\infty a^{b+1} e^{-\frac{1}{2}\frac{a^2}{\sigma_A{}^2}} da \tag{1}$$

This is identical to equation (9) of appendix I.

IT is shown in Appendix II that $E[D(t)]$ defined by equation (22) of Chapter III becomes an exact expression for the rate of accumulation of damage when the means between successive extrema on any record are zero. Because a narrow-band process is a sinusoid with slowly varying phase and amplitude, the means between successive extreme will be approximately zero. Therefore, for a narrow-band stress process it should be expected that equation (9) of Appendix I would agree with equation (22) of Chapter III.

---

*Miles did not actually derive an expected rate of accumulation of damage. He found an equivalent sinusoidal stress with which a fatigue life calculation can be made. His calculation can be transformed exactly into equation (9), however.

Comparison of the Method of Calculating Fatigue Damage Developed in Chapter III with the Peak Stress Method

A method which is quite often mentioned, [9] and [20], as a possible approach for assessing damage due to complex stress records and random stress processes is the peak stress method. This method associates with each positive peak or negative trough the damage that would be done by one half cycle of a sinusoid with the same amplitude as the positive peak or negative trough. Positive troughs and negative peaks are neglected.

To see how this method compares with the method proposed in Chapter III, consider a record which has no positive troughs or negative peaks. Then, for a portion of this record which has extrema at $t_1, \ldots, t_k$, positive peaks and negative troughs only, the damage calculation by the peak stress method is

$$\text{damage}_{\text{p.s.}} = \frac{1}{2C}\left(|a^i(t_1)|^b + |a^i(t_2)|^b + \ldots + |a^i(t_k)|^b \right) \qquad (2)$$

By the definition of $f(x)$ in Chapter III, the damage calculated by the method proposed in Chapter III is exactly this sum. Therefore, for records of this type the peak stress method and the method proposed in Chapter III agree. The sample functions of a narrow-band random process are of this type. For records which have many positive troughs or negative peaks the peak stress method can give a nonrealistic estimate of the damage while the method proposed in Chapter III gives a reasonable estimate. Consider a record composed of all negative peaks and troughs, all about the same magnitude. Let $t_1, \ldots, t_{2k}$ be the times at which the above extrema occur where even subscripts correspond to troughs. Then the damage calculated by the peak stress method is

$$\text{damage}_{\text{p.s.}} = \frac{1}{2C} \quad |a^i(t_2)|^b + |a^i(t_4)|^b + \ldots + |a^i(t_{2k})|^b \qquad (3)$$

The actual damage for such a record is approximately equal to the damage calculated by the half-wave design method. Since this estimate is always less than the estimate given by the method proposed in Chapter III and since the method proposed in Chapter III assesses a very small amount of damage to the type of record being considered here, the actual damage and both of these estimates are very small; however, if the magnitudes of the $a^i(t_k)$'s are large the peak stress method associates a very large, thus unreasonable, amount of damage with this type of record.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS

Conclusions

For the reasons that follow, the method of calculating fatigue damage developed in Chapter III, based on the damage function $f(a^i(t))$, should give a reasonable estimate of the actual damage accumulated for a stress record which is a sample function of a random process.

1. The discussions in Appendix II and Chapter IV show that a very favorable comparison exists between the damage function method of assessing fatigue damage for complex stress records and both the half-wave design method and the peak stress method.
2. The data taken by Starky and Marco shows that this method does not give a bad estimate of the actual damage for the complex stresses that they tested, Fig. 8. Appendix III.
3. In the case of a narrow-band normal stress process the method of calculating fatigue damage developed in Chapter III for an arbitrary stress process agrees well with the method developed by Miles for calculating fatigue damage for a narrow-band normal stress process.

Recommendations

While the method of calculating fatigue damage derived in Chapter III agrees respectably with previous work, this in itself does not constitute complete verification of the use of this method in calculating expected fatigue damage for wide-band random stress processes. The only way to make such a verification would be to perform extensive fatigue tests where the power spectrum of the stress process could be varied. Certainly, future work should be aimed in this direction.

It is shown in Appendix II, that for compression means between successive extrema the damage function calculation severely over-estimates the fatigue damage as calculated by the half-wave design method. Since there is no evidence to indicate how valid the half-wave design method is for a wide-band random stress process, exactly how much the damage function method for calculating fatigue life over-estimates the actual damage incurred in such a process cannot be determined. An experimental investigation, such as indicated above, might show that the damage function calculation could be reduced by some amount in order to adjust for this fact.

There is a basic concept in the development of the damage function which might prove interesting in future work. This is that the damage function was an initial attempt to rewrite the cumulative damage concept in terms of the stress and its first derivative rather than in terms of peaks and means, etc. A refined theory of fatigue written directly in terms of such basic quantities would be very desirable, since it would allow direct application to both derterministic and random stresses.

APPENDIX I

THE NARROW-BAND FATIGUE PROBLEM

The method of calculating fatigue damage for a narrow-band random stress process is a standard result and is widely used in the aircraft industry. The purpose of the following work is to briefly discuss the procedure used for obtaining this result with particular emphasis on how the narrow-band assumption is employed.

Method of Analysis

A narrow-band random process $\{y(t)\}$, is a random process which can be represented as the real part of the expression $\{A(t)\, e^{(i\, w_o t + i\phi - i\,\Phi(t))}\}$ where $\{A(t)\}$, the peak amplitude, and $\{\Phi(t)\}$, the phase change, are random processes whose records vary slowly in comparison with $\sin w_o t$ and where ϕ is a random variable uniformly distributed over the interval $(0, 2\pi)$.* Any sample function of such a process will appear as a sinusoid with slowly varying amplitude and phase. Because of this sinusoidal property, if a positive peak of some particular level a occurs in any record, a cycle of level a occurs. Then by knowing the expected frequency and the distribution of positive amplitudes equation (1) or (4) of Chapter II can be used to find the expected rate of accumulation of damage. For a normal stress process which is ergodic, has a zero mean, and is narrow-band both the expected frequency and the distribution of peak amplitudes can be found if the autocorrelation function is known. Finding these quantities for a normal

* See Middleton [15] p. 158.

process is a special case. In general it would require more than just the knowledge of the autocorrelation function to determine the quantities.

Expected Frequency

Consider any sample function $y^i(t)$ of a random process $\{Y(t)\}$. Assume the function is differentiable. Let $u(x) = (1$ if $x \geq 0$, or 0 if $x < 0)$. Every time $y^i(t)$ changes from negative to positive there is a unit positive change in $u(y^i(t))$. Every time $y^i(t)$ changes from positive to negative there is a unit negative change in $u(y^i(t))$.

The sum of the positive changes plus the sum of the negative changes is the total number of zero crossings in a time interval (T_1, T_2). Formally differentiating $u(y^i(t))$ with respect to time yields,

$$\frac{d}{dt}\left[u(y(t))\right] = \dot{y}^i(t)\,\delta\left[y^i(t)\right]^* \tag{1}$$

Then the sum of positive changes can be written as,

$$\Sigma(\text{positive changes}) = \int_{(T_1,\ T_2 \text{ where } \dot{y}(t)>0} \dot{y}^i(t)\,\delta(y^i(t))\,dt \tag{2a}$$

and the sum of negative changes can be written as,

$$\Sigma(\text{negative changes}) = -\int_{(T_1,\ T_2) \text{ where } \dot{y}(t)<0} \dot{y}^i(t)\,\delta(y^i(t))\,dt \tag{2b}$$

Therefore, the number of zero crossing $N^i(0, T_2 - T_1)$ is

$$N^i(0,\ T_2 - T_1) = \int_{T_1}^{T_2} |\dot{y}^i(t)|\,\delta(y^i(t))\,dt \tag{2c}$$

$^*\delta(x)$ is the Dirac delta function.

The number of zero crossings per unit time $n^i(0,t)$ is

$$n^i(0,t) = |\dot{y}^i(t)|\delta(y^i(t)) \tag{3}$$

If the process is differentiable in the man-square sense, the expected number of zero crossings per unit time is

$$E[n(0,t)] = \iint_{\infty}^{\infty} |\dot{y}|\delta(y)\, f_{\dot{y}y}(\dot{y},y;t)\,d\dot{y}\,dy \tag{4}$$

Since a stationary normal process was assumed at the beginning, the density function $f_{\dot{y}y}(y, \dot{y};t) = \dfrac{1}{2\pi\sigma_y\sigma_{\dot{y}}} \times e^{-\frac{1}{2}\left(\frac{y^2}{\sigma_y^2}+\frac{\dot{y}^2}{\sigma_{\dot{y}}^2}\right)}$.*

Evaluating equation (4) using this density function, $E[n(0,t)]$ is $\dfrac{\sigma_{\dot{y}}}{\pi\sigma_y}$.

If the process is narrow-band, the expected frequency $w_{o/2\pi}$ is the number of zero crossings with positive slope. This will be half the number of zero crossings.**

$$w_{o/2\pi} = \frac{1}{2\pi}\,\frac{\sigma_{\dot{y}}}{\sigma_y} \tag{5}$$

Distribution of Positive Peak Amplitudes

By changing $\delta(y(t))$ to $\delta(y(t) - y_o)$ the above analysis can be used to obtain the expected number of crossings per unit time of a level a with positive slope. This is

$$E[n(a+, t)] = \frac{1}{2\pi}\,\frac{\sigma_{\dot{y}}}{\sigma_y}\,e^{-\frac{a^2}{2\sigma_y^2}} \tag{6}$$

* This density function is the same as equation (20) in Chapter III, where the same type of process was considered.

**This derivation is almost identical to the one Middleton [15] presents in Section 9.4.

A positive peak of level a occurs in a record when it crosses the level a with positive slope and does not cross the level a + da. A positive trough occurs in a record when it crosses the level a + da with positive slope but does not cross the level a. Thus, the expected number of crossings of the level a with positive slope per unit time $E\,[\eta(a^+,\ t)]$ equals the expected number of peaks per unit time plus the expected number of crossings of the levels a and a + da with positive slope per unit time. The expected number of crossings of the level a + da with positive slope per unit time $E\,[\eta(a{+}da^+, t)]$ equals the expected number of troughs of level a plus the expected crossings of the levels a and a + da with positive slope per unit time. Thus the expected number of peaks minus troughs of level a per unit time equals $E[\eta(a^+,t)] - E[\eta(a + da^+,t)]$. When considering a narrow-band process the occurrence of a positive trough is almost impossible. Therefore, the expected number of positive peaks of level a per unit time is $E[\eta(a^+,t)] - E[\eta(a{+}da^+,t)]$. Let $f_{p^+}(a)da$ be the probability of occurrence of a peak of level a. Since there is exactly one positive peak for each cycle of a record of a narrow-band process, $E[\eta(0^+,t)]\, f_{p^+}(a)da$ is the expected number of positive peaks of a level a per unit time. Therefore,

$$E[\eta(0^+,t)]\ \ f_{p^+}(a)da = E[\eta(a^*,t)] - E\,[\eta(a{+}da^+,t)] \tag{7}$$

$$= -\,\frac{dE[\eta(a^+,t)]}{da}\,da$$

Substituting (6) and (7), the distribution for positive peaks is

$$f_p + (a) = \frac{a}{\sigma_y{}^2}\, e^{-\frac{a^2}{2\sigma_y{}^2}} \tag{8}$$

This derivation is due to A. Powell [21].

The Damage

Substituting the distribution of peak amplitudes and the expected frequency into equation (4) of Chapter II, the expected rate of accumulation of damage is

$$E[\dot{D}(t)] = \frac{\sigma_{\dot{A}}}{2\pi \sigma_A \sigma_A^2} \int_0^\infty a^{b+1} e^{-\frac{a^2}{2\sigma_y^2}} da \tag{9}$$

APPENDIX II

DISCUSSION OF THE PROPERTIES OF THE DAMAGE FUNCTIONAL MENTIONED ON PAGE 23

The Derivative of $f(a^i(t))$

Assume that the original stress records are differentiable with bounded derivatives. Note that $f(x)$ as defined by equation (7) of Chapter III is $f(x) = \begin{cases} x^b, & x>0 \\ -|x|^b, & x<0 \end{cases}$ where $b>1$. For $x>0$ or $x<0$, this function has a derivative, $\frac{d\ f(x)}{dx} = b|x|^{b-1}$. For $x = 0$, the left-hand derivative, $\lim_{x \to 0} - \frac{-|x|^b - 0}{-|x| - 0}$, is zero, and the right-hand derivative, $\lim_{x \to 0^+} \frac{x^b - 0}{x - 0}$, is zero. Hence, $f(x)$ has a derivative for all x. Therefore, by the chain rule, the derivative of $f(a^i(t))$ is $b|a^i(t)|^{b-1} \frac{da^i(t)}{dt}$.

Extreme Values of $f(a^i(t))$

The second property is that $f(a^i(t))$ takes on extreme values for the same values of t that $a^i(t)$ takes on extreme values. This is obvious from examining the definition of $f(x)$. Note that $f(x)$ is an increasing function of x. For any t_o where $a^i(t)$ is not an extremum of $a^i(t)$, there must be some neighborhood of t_o so that $a^i(t)$ is either increasing or decreasing in this neighborhood. Therefore, $f(a^i(t))$ must be either increasing or decreasing in this neighborhood, and hence $f(a^i(t_o)$ cannot be an extremum of $f(a^i(t))$. If $a^i(t_o)$ is a maximum of $a^i(t)$, there is some neighborhood of t_o so that for any t^1 in this neighborhood $a^i(t^1)$ is less than $a(t_o)$. Therefore, $f(a^i(t^1)$must be less than $f(a^i(t_o))$. Hence, $f(a^i(t_o))$must be a maximum of $f\ (a^i(t))$. A similar argument applies for minimums.

The Relation Between the Damage Functional and The Half-Wave Design Method

The bases for the relation are inequalities (12) and (13) of Chapter III.

Inequality (12)

Inequality (12) applies only for a compression or zero mean between successive extrema, and therefore, it is considered only for the case where $a_{m_n} < 0$. There are three subcases of this inequality:

1. $a_{m_n} < 0$ and $0 > a^i(t_n) > a^i(t_{n+1})$
2. $a_{m_n} < 0$ and $a^i(t_n) > 0 > a^i(t_{n+1})$*
3. $a_{m_n} = 0$

Subcase 1. Since a_{m_n} is the mean value between the peaks, $a^i(t_n)$ and $a^i(t_{n+1})$, a_{m_n} can be replaced by $\frac{a^i(t_n) + a^i(t_{n+1})}{2}$. Then, (12) of Chapter III becomes

$$\left| f\,(a^i(t_n)) - f\,(a^i(t_{n+1}))\right| \geq 2\left|\frac{a^i(t_n) - a^i(t_{n+1})}{2}\right|^b \tag{1}$$

By the definition of the function f(x),

$$\left| f\,(a^i(t_n)) - f(a^i(t_{n+1}))\right| = \left| a^i(t_{n+1})\right|^b - \left|a^i(t_n)\right|^b \tag{2}$$

Therefore, inequality (1) becomes

$$\left|a^i(t_{n+1})\right|^b - \left|a^i(t_n)\right|^b \geq 2\left|\frac{a^i(t_{n+1}) - a^i(t_n)}{2}\right|^b \quad ** \tag{3}$$

*$0 > a^i(t_{n+1}) > a^i(t_n)$ and $a^i(t_{n+1}) > 0 > a^i(t_n)$ are also subcases but because of the absolute value signs on both sides of (12), the same proof holds.

**The absolute value signs on the interior differences on the right-hand side do not change the value from (1) to (3), since $0 > a^i(t_n)$ and $0 > a^i(t_{n+1})$.

Dividing through by $|a^i(t_{n+1})|$ and noting that $|a^i(t_{n+1})| > a^i(t_n)|$ for subcase 1, the inequality (3) becomes

$$1-x^b \geq 2(\frac{1-x}{2})^b \tag{4}$$

where $0<x = |\frac{a^i(t_n)}{a^i(t_{n+1})}|<1$. Since $0 < x < 1$, $x^b < x$. Then, $1-x^b > 1-x$, and $0 < \log(1-x^b) > \log(1-x)$.

But since $b > 1$, $0 > \log(1-x) > b \log(1-x)$. Therefore, $0 > \log(1-x^b) > b \log(1-x)$, or $(1-x^b) > (1-x)^b$. Since 2^b 2, $2/2^b < 1$.

Therefore,

$$1 - x^b > 2 \left|\frac{1-x}{2}\right|^b \tag{5}$$

<u>Subcase 2.</u> Using the fact that a_{mn} is the mean, inequality (12) of Chapter III can again be written in the form of inequality (1). Since for this subcase $a^i(t_{n+1})$ and $a^i(t_n)$ have opposite signs, because of the definition of f( ), inequality (1) becomes

$$|a^i(t_{n+1})|^b + |a^i(t_n)|^b > 2 \left|\frac{|a^i(t_{n+1})| + |a^i(t_n)|}{2}\right|^b \tag{6}$$

By dividing through by $a^i(t_{n+1})^b$, inequality (6) becomes

$$1 + x^b > 2\left|\frac{1+x}{2}\right|^b \tag{7}$$

where $0<x = \left|\frac{a^it_n}{a^i(t_{n+1})}\right|<1$. Since $f(x) = x^b$, $b>1$, is a convex function for $x>0$,

$$x_1^b + x_2^b > 2(\frac{x_1 + x_2}{2})^b \tag{8}$$

for x_1 and $x_2>0$ [22]. This proves subcase 2 by letting $x_1=1$ and $x_2=x$ in (7).

Subcase 3. This is obvious. When $a_{m_n}=0$, $-a^i(t_{n+1}) = a^i(t_n)$, and inequality (1) becomes

$$2\left|a^i(t_{n+1})\right|^b = 2\left|\frac{2a^i(t_{n+1})}{2}\right|^b \tag{9}$$

Inequality (13)

Inequality (13) of Chapter III applies only for tension means, and therefore it is considered only for the case where $a_{m_n}>0$. There are two subcases of this inequality:

1. $a_{m_n}>0$, $a^i(t_n)> a^i(t_{n+1})> 0$
2. $a_{m_n}>0$, $a^i(t_n) > 0 > a^i(t_{n+1})$*

Subcase 1. Again, a_{m_n} can be replaced by $\dfrac{a^i(t_n)+a^i(t_{n+1})}{2}$, and inequality (13) of Chapter III becomes

$$\left|f(a^i(t_n)) - f(a^i(t_{n+1}))\right| \geq 2\left|\frac{a^i(t_n) - a^i(t_{n+1})}{2\left\{1-\dfrac{a^i(t_n)+a^i(t_{n+1})}{2a_u}\right\}}\right| \tag{10}$$

Using the definition of f(x), inequality (10) becomes

$$a^i(t_n)^b - a^i(t_{n+1})^b \geq 2\left[\frac{a^i(t_n) - a^i(t_{n+1})}{2\left\{1-\dfrac{a^i(t_n)+a^i(t_{n+1})}{2a_u}\right\}}\right]^{**} \tag{11}$$

Letting $x = \dfrac{a^i(t_{n+1})}{a^i(t_n)}$ and $\alpha = \dfrac{a^i(t_n)}{a_u}$ a function F(x) can be defined as

$$F(x) = 1-x^b - 2\left[\frac{1-x}{2}\right]^b \times \left[\frac{1}{1-\frac{\alpha}{2}(1+x)}\right]^b \quad \begin{matrix}\alpha \text{ in}(0,1) \\ x \text{ in}(0,1)\end{matrix} \tag{12}$$

*See footnote on page

**No absolute value signs needed here since $a^i(t_n) > a^i(t_{n+1}) \geq 0$.

If $F(x) \geq 0$, inequality (13) of Chapter III will hold. For $x = 0$, $F(x)$ will be negative unless α is restricted to the interval $(0,\ 2(-1/2^{\frac{b-1}{b}}))$. From the graph of $F(x)$ in Fig. 6 it is seen that $F(x) \geq 0$ for x in $(0,1)$ and for any α in $(0,\ 2(1 - 1/2^{\frac{b-1}{b}}))$. Therefore, inequality (13) is true for this subcase provided α is in $(0,\ 2(1-1/2^{\frac{b-1}{b}}))$.

To see what restriction this places on the stress records, note that by definition $\alpha = \frac{a^i(t_n)}{a_u}$. This is the ratio of the larger of the two extrema to the ultimate strength of the material. Since the material constant b takes on a value between five and twenty for most structural materials [6], $2(1 - 1/2^{\frac{b-1}{b}})$ will be greater than 0.8. The ratio betwen the yield point stress, $a_{y.p}$, and the ultimate stress for most materials is also about 0.8. Then the restriction, $\frac{a^i(t_n)}{a_u} \leq 2(1 - 1/2^{\frac{b-1}{b}})$, is no more limiting than the reasonable design criteria, that the yield point stress should not be exceeded.

<u>Subcase 2.</u> Replacing a_{m_n} by $\frac{a^i(t_n) + a^i(t_{n+1})}{2}$ and using the definition of $f(x)$, inequality (13) of Chapter III becomes

$$|a^i(t_n)|^b + |a^i(t_{n+1})|^b \leq 2 \left[\frac{|a^i(t_n)| + |a^i(t_{n+1})|}{2\{1 - \frac{a^i(t_n) + a^i(t_{n+1})}{2a_u}\}}\right]^b \tag{13}$$

Making the substitutions, $x = \left|\frac{a^i(t_{n+1})}{a^i(t_n)}\right|$ and $\alpha = \frac{a^i(t_n)}{a_u}$, a function $G(x)$ can be formed,

$$G(x) = 1 + x^b - 2\left(\frac{1+x}{2}\right)^b \left[\frac{1}{1 - \frac{\alpha}{2}(1-x)}\right]^b \qquad \begin{matrix}\alpha \text{ in } (0,\ 1) \\ x \text{ in } (0,\ 1)\end{matrix} \tag{14}$$

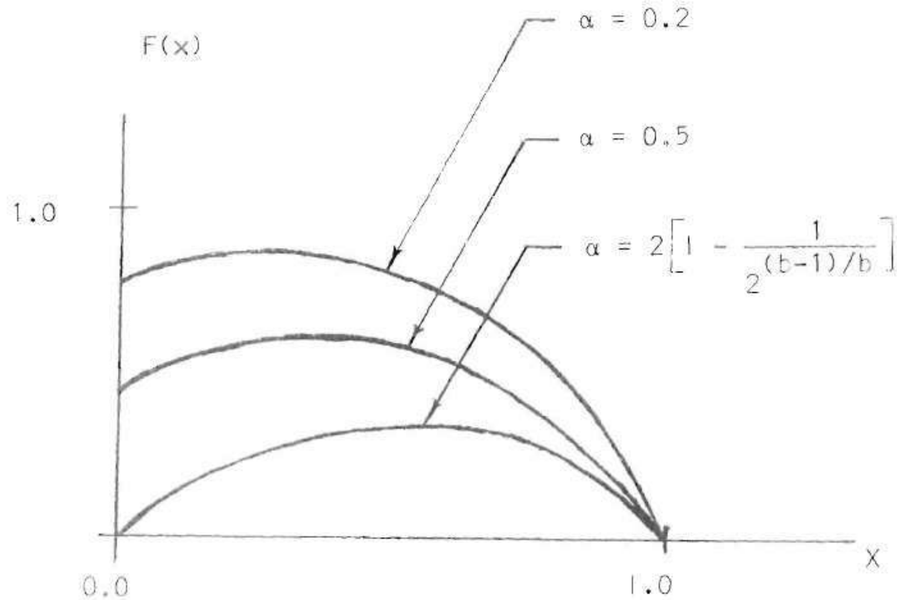


Figure 6. Graph of F(x) for Various Values of α and a Value b > 5.

If $G(x) \geq 0$, inequality (13) of Chapter III will hold. For $x=0$, $G(x)$ will be negative unless α is restricted to the interval $(0,\ 2(1-1/2^{\frac{b-1}{b}}))$. Differentiating $G(x)$, $\left.\frac{d}{dx}G(x)\right|_{x=1}$ is

$$\left.\frac{d}{dx}G(x)\right|_{x=1} = +b \tag{15}$$

Since $G(x) = 0$ for $x=1$, $G(x)$ must be negative for at least some x close to 1. Therefore inequality (13) of Chapter III does not hold for all x in$(0,1)$. The graphs of $G(x)$ in Fig. 7., for values of the material constant, b = 5.0, 8.0, 12.0 and 15.0,* show that for α as high as 0.5 the function $G(x)$ is negative only for x in$(0.55,\ 1.0)$ and the lowest value it takes on is 0.2. By substituting $x = \left|\frac{a^i(t_{n+1})}{a^i(t_n)}\right|$ and $\alpha = \frac{a(t_n)}{a_u}$ into $G(x)$ and multiplying by $|a^i(t_n)|^b$, the expression,

$$|a^i(t_n)|^b \cdot G(x) = |a^i(t_n)|^b + |a^i(t_{n+1})|^b - 2\left|\frac{a^i(t_n) + a^i(t_{n+1})}{2\left[1 - \frac{a^i(t_n + a^i(t_{n+1})}{2a_u}\right]}\right|^b \tag{16}$$

is obtained. The right-hand side of this equation is the difference between the calculation of fatigue damage by the method proposed in Chapter III and the half-wave design method calculation. Therefore $a^i(t_n)^b \times G(x)$ equals this difference. The fatigue assessed by the damage function is $a^i(t_n)^b + a^i(t_n)^b$. This quantity is greater than $a^i(t_n)^b$. For $\alpha = 0.5$ and for the worst possible value of the ratio, x, between the two successive extrema, it can be seen from equation (16) that the damage assessed by the damage function method will be more than

*These values are representative of the range of the constant b, for different structural materials.

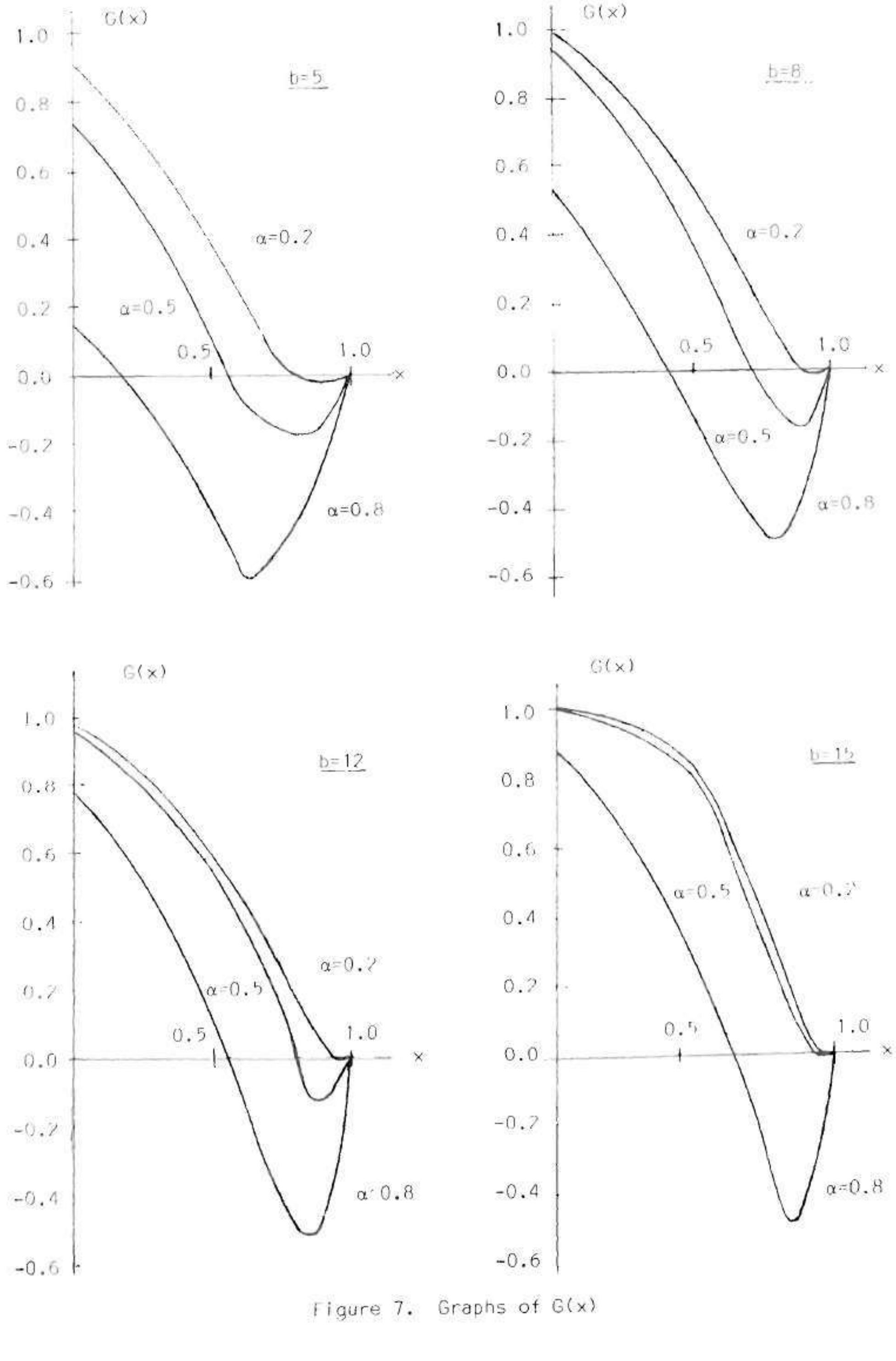


Figure 7. Graphs of G(x)

83 percent of the damage assessed by the half-wave-design method.

Therefore, for any stress record which has few positive peaks above half the ultimate strength of the material, about 63 percent of the yield point stress,* the relation between the damage function and the half-wave design method is at least approximately true. While this restriction is more limiting than the restriction in the first subcase, it still leaves a wide range of application.

---

*$a_{y.p.}$ is about 8/10 of a_u. See discussion of other subcase.

APPENDIX III

COMPARISON OF THE COMPLEX STRESS DATA OF STARKY AND MARCO WITH THE FATIGUE CALCULATION DISCUSSED IN CHAPTER III

To varify the half-wave design method Starky and Marco performed experiments on a complex stress fatigue testing machine 12 . This machine was capable of producing stress records of the form

$$a(t) = A_1(\sin x + 1/M \sin(2x-\phi) \tag{1}$$

x is $w_o t$ where w_o is 100 radians per second. A_1 is the amplitude of the low frequency component. M is the amplitude ratio, A_{1/A_2}, where A_2 is the amplitude of the second component of frequency $2w_o$; and ϕ is the phase angle of the second component relative to the first. Tests were performed for M values of 1/10, 1/4,. 2/3, 4/3, and 2 and ϕ values of 0 and $\pi/2$. For each M and ϕ used, the machine was controlled for various values of peak stress, and the fatigue life was measured in cycles of the period of the complex wave formed.

The graphs in Fig. 8 are a reproduction of the graphs made by Starky and Marco to show the results of their experiment for the test on aluminum. The circles represent the data. Curves 4 and 1 represent the half-wave design method, using a refinement of the concept of cumulative damage and using Miner's concept of cumulative damage, respectively. Curves 2 and 3 represent calculations of fatigue life neglecting secondary peaks. The range of stress conversion used for 2 is the Goodman and for 3 is the Gerber. Curve 5 is superimposed on the work of Starky and Marco to show how the damage function calculation of fatigue life compares with the other calculations and the data. Note that in every case the damage function calculation allows the shortest fatigue life and in some cases is considerably below the data.

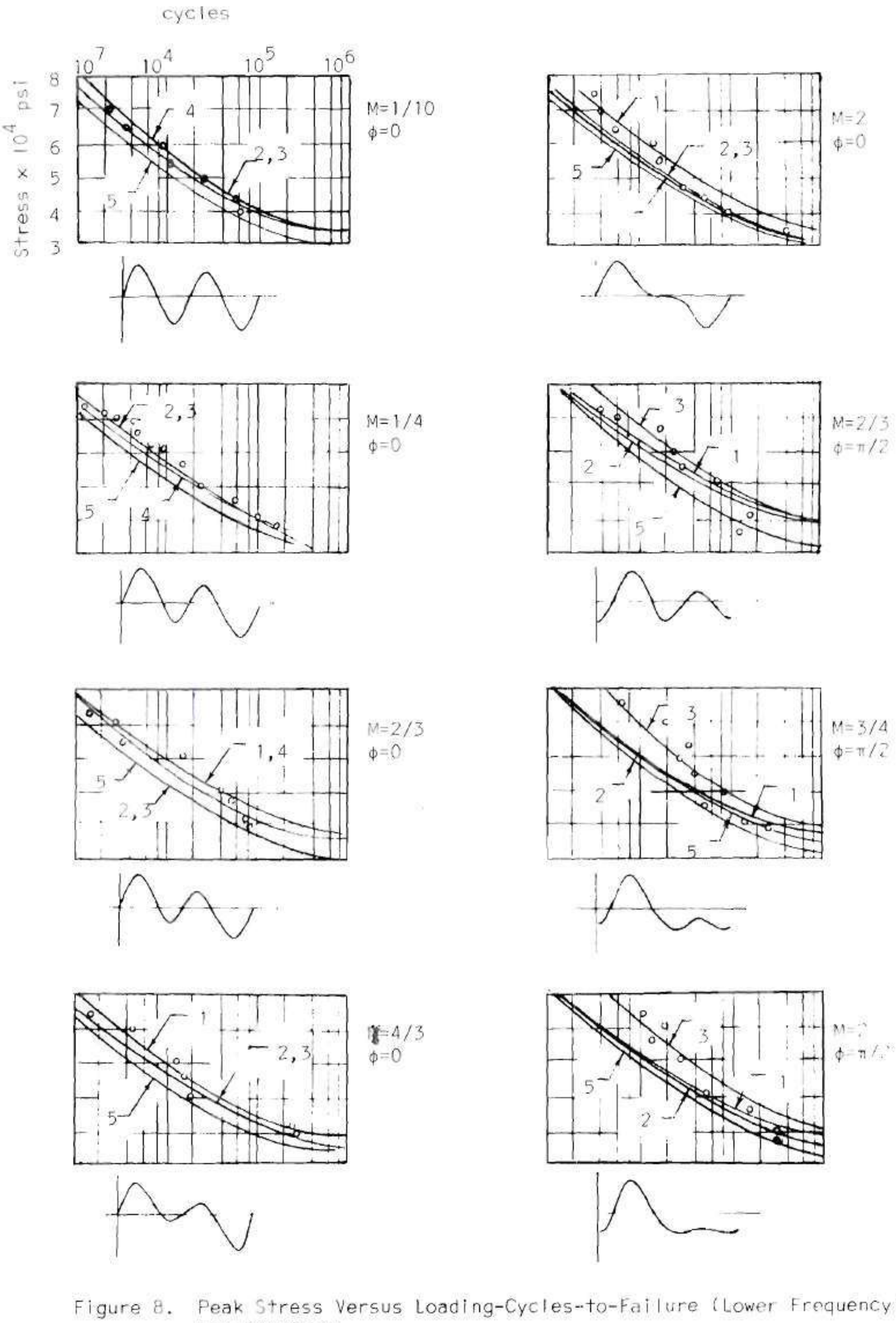


Figure 8. Peak Stress Versus Loading-Cycles-to-Failure (Lower Frequency), For Aluminum.

LITERATURE CITED

1. H. J. Grough, The Fatigue of Metals, Ernest Benn Limited, London, 1926, Chapter IX.

2. T. J. Dolan, B. J. Lazan, O. J. Horger, Fatigue, American Society for Materials, Cleveland, Ohio, 1953.

3. B. F. Langer, "Fatigue Failure From Stress Cycles of Varying Amplitude," Journal of Applied Mechanics, American Society of Mechanical Engineers, Vol. 4, 1937, A-160.

4. M. A. Miner, "Cumulative Damage in Fatigue," Journal of Applied Mechanics, American Society of Mechanical Engineers, Vol. 12, 1945, A-159.

5. Trapp and Lazan, "Material Properties That Affect Fatigue Life, and the Role of Damping," Symposium on Acoustical Fatigue, American Society for Testing Materials, 1960, p. 3.

6. S. H. Crandall and W. D. Mark, Random Vibration in Mechanical Systems, Academic Press, 1963.

7. J. W. Miles, "On Structural Fatigue Under Random Loading," Journal of Aerospace Science, Vol. 21, 1954, p. 753.

8. W. D. Mark, "The Inherent Variation in Fatigue Damage Resulting from Random Vibration," Ph.D. Thesis, Department of Mechanical Engineering, M.I.T., August 1961.

9. F. A. McClintock, "Fatigue of Metals," Random Vibration, (Ed. S. H. Crandall, M.I.T. Press 1958.

10. A. R. Wade and P. Groutenhuis, "Very High Speed Fatigue Testing," International Conference on Fatigue of Metals, The Institution of Mechanical Engineers, London, 1956, p. 361.

11. T. W. Lomas, J. W. Ward, J. R. Rait, E. W. Colbeck, "The Influence of Frequency of Vibration on the Endurance Limit of Ferrous Alloys at Speeds Up to 150,000 Cycles per Minute Using a Pneumatic Resonance System," International Conference on Fatigue of Metals, The Institution of Mechanical Engineers, London, 1956, p. 375.

12. W. L. Starky and S. M. Marco, "Effect of Complex Stress Time Cycles on Fatigue Properties of Materials," Transactions of the American Society of Mechanical Engineers, Vol. 79, 1957, p. 1329.

13. H. C. O'Connor, J. L. Morrison, and M. I. Mech, "The Effect of Mean Stress on Push-Pull Fatigue Properties of Alloy Steel," International Conference on Fatigue of Metals, The Institution of Mechanical Engineers, London, 1956, p. 102.

14. J. E. Moyal, "Stochastic Processes and Statistical Physics," Symposium on Stochastic Processes, Royal Statistical Society, 1945, p. 150.

15. D. Middleton, Introduction to Statistical Communication Theory, McGraw-Hill, 1960.

16. T. M. Apostol, Mathematical Analysis, Addison-Wesley, 1957, Ch. 8.

17. Laning and Battin, Random Processes in Automatic Control, McGraw-Hill, 1956, p. 120.

18. Powell, "On The Response of Structures to Random Pressure and Jet Noise in Particular," Random Vibration (Ed. S. H. Crandall), M.I.T. Press 1958.

19. A. J. F. Siegert, "Passage of Stationary Processes Through Linear and Non-Linear Devices," I.R.E. Transactions on Information Theory, P.G.I.T. 3, March 1954.

20. A. K. Head and F. H. Hooke, "Random Noise Fatigue Testing," International Conference on Fatigue of Metals, The Institute of Mechanical Engineers, London, 1956, p. 301.

21. Powell, "On Fatigue Failure of Structures Due to Vibration Excited by Random Pressure Fields," Journal of the Acoustical Society of America, Vol. 30, 1958, p. 1130.

22. R. P. Boas, Jr., A Primer of Real Functions, The Mathematical Association of America, 1960.